

Isotropic self-similar Markov processes

Ming Liao¹ and Longmin Wang²

Summary We show that an isotropic self-similar Markov process in \mathbb{R}^d has a skew product structure if and only if its radial and angular parts do not jump at the same time.

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1 Introduction and main result

It is well known that a Brownian motion in \mathbb{R}^d ($d \geq 2$) has a skew product structure, that is, it may be expressed as a product of its radial process and a time changed spherical Brownian motion. Moreover, the radial process is a Bessel process and is independent of the spherical Brownian motion, and the time change is adapted to the radial process. This decomposition is naturally related to the invariance of the Brownian motion under the group $O(d)$ of orthogonal transformations on \mathbb{R}^d . More generally, Galmarino [3] proved that a continuous isotropic or $O(d)$ -invariant Markov process in \mathbb{R}^d is also a skew product of its radial motion and an independent spherical Brownian motion with a time change. Pauwels and Rogers [11] and Liao [10] extended these results to more general settings.

Because any continuous isotropic Markov process has a skew product structure, it is therefore natural to consider a similar skew product for discontinuous isotropic Markov processes. Graversen and Vuolle-Apiala [4] discussed a skew product for isotropic α -self-similar Markov processes, which include the purely discontinuous symmetric $(1/\alpha)$ -stable processes. Their main result says that after a time change due to Lamperti [8] and Kiu [7], the radial process and the angular process are respectively multiplicatively invariant and $O(d)$ -invariant Markov processes, and are independent. This leads to a skew product structure similar to that of a Brownian motion. However, as will be shown later, the independence part of this interesting result holds only under a rather restrictive condition, which excludes for example the symmetric $(1/\alpha)$ -stable processes for $\alpha > 1/2$. We note that the proof of Proposition 2.4 in [4] has an error in the conditional expectation argument.

The aim of this paper is to clarify this rather important point. We will show that an isotropic α -self-similar Markov process has a skew product structure if and only if its radial and angular parts do not jump at the same time.

¹Department of Mathematics, Auburn University, Auburn, AL 36849, USA. Email: liaomin@auburn.edu

²School of Mathematical Sciences, Nankai University, Tianjin, China. Email: wanglm@nankai.edu.cn

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After the completion of this paper, we found the independence error in [4] has been noted in Bertoin and Werner [1]. They used the correct part of [4] in their work, but did not pursue the independence problem.

We now describe our setup more precisely. All processes considered in this paper are assumed to have càdlàg paths (right continuous paths with left limits). Let x_t be a (time homogeneous) Markov process in \mathbb{R}^d , $d \geq 2$, with transition function P_t satisfying the usual simple Markov property. We will allow process x_t to have a possibly finite life time, and as usual let P_x denote the distribution of process starting at x on the canonical space of càdlàg paths with possibly finite life times.

The restriction of process x_t on $E = \mathbb{R}^d - \{0\}$, defined before reaching the hitting time of origin 0, is also a Markov process. For simplicity, this process together with its transition function and distribution are still denoted by x_t , P_t and P_x .

The process x_t in \mathbb{R}^d is said to be *isotropic* or *$O(d)$ -invariant* if

$$P_t(\phi(x), \phi(B)) = P_t(x, B) \quad (1)$$

for any $\phi \in O(d)$, $x \in \mathbb{R}^d$ and Borel subset $B \subset \mathbb{R}^d$. This is equivalent to saying that process $\phi(x_t)$ with $x_0 = x$ has the same distribution as process x_t with $x_0 = \phi(x)$.

The process x_t is said to be *α -self-similar*, or *α -s.s.* in short, for some constant $\alpha > 0$, if

$$P_{\lambda t}(x, B) = P_t(\lambda^{-\alpha}x, \lambda^{-\alpha}B) \quad (2)$$

for any $\lambda > 0$, $x \in \mathbb{R}^d$ and $B \subset \mathbb{R}^d$. This is equivalent to saying that process $x_{\lambda t}$ with $x_0 = x$ has the same distribution as process $\lambda^\alpha x_t$ with $x_0 = \lambda^{-\alpha}x$.

It is clear that if x_t is $O(d)$ -invariant and/or α -self-similar, so is its restriction to E .

In what follows, we will exclusively consider an isotropic Markov process x_t in E . For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we denote by $|x| = \sqrt{x_1^2 + \dots + x_d^2}$ the radial part of x . Let $r_t = |x_t|$ and $\theta_t = x_t/r_t$ be the radial and angular parts of x_t . It is easy to show using the $O(d)$ -invariance (see for example [10]) that $r_t > 0$ is a 1-dim Markov process with transition function R_t given by

$$R_t f(r) = P_t(f \circ \pi_1)(x)$$

for $r > 0$ and Borel function f on $(0, \infty)$, where $\pi_1: E \rightarrow (0, \infty)$ is the natural projection given by $y \mapsto |y|$ and x is any point of E with $r = |x|$. As the angular process θ_t lives in the unit sphere S^{d-1} which is invariant under the action of $O(d)$, one would expect that it should inherit the $O(d)$ -invariance of x_t in some sense. This leads to the following definition of a skew product structure.

Definition Let x_t be an isotropic Markov process in E . We say that x_t has a *skew product structure* if $x_t = r_t \xi_{A_t}$, where A_t is a continuous and strictly increasing process with $A_0 = 0$,

adapted to the radial process r_t , and ξ_t is an $O(d)$ -invariant Markov process in S^{d-1} and is independent of process r_t .

Because $O(d)$ acts on S^{d-1} transitively, S^{d-1} may be regarded as a homogeneous space of $O(d)$. Invariant Markov processes in homogeneous spaces are Feller processes, and their generators may be expressed explicitly in terms of an invariant differential operator and a Lévy measure (see Section 3 for more details), thus providing a useful tool for studying these processes.

The following is our main theorem.

Theorem 1 *Let x_t be an isotropic α -self-similar Markov process in $E = \mathbb{R}^d - \{0\}$ ($d \geq 2$). Then x_t has a skew product structure if and only if its radial and angular parts do not jump at the same time, that is, for all $x \in E$, P_x -almost surely, r_t and θ_t cannot jump together at same time t for any $t \geq 0$.*

Proof of the necessity part The sufficiency of the condition will be proved in Section 3. For the necessity, we assume that x_t has the skew product $x_t = r_t \xi_{A_t}$. Since r_t is càdlàg, by [6, Proposition I.1.32], the random set $\{\Delta r_t \neq 0\}$ is thin in the sense that there is a sequence of stopping times τ_n such that $\{\Delta r_t \neq 0\} = \bigcup_n [[\tau_n]]$, where $\Delta r_t = r_t - r_{t-}$, and $[[\tau_n]]$ is the graph of τ_n , i.e., $[[\tau_n]] = \{(\omega, t), t \in \mathbb{R}_+, t = \tau_n(\omega)\}$. For any $n \geq 1$, the time A_{τ_n} is measurable in process r_t , and the independence of r_t and ξ_t implies that A_{τ_n} is independent of ξ_t . As a Feller process, ξ_t is quasi-left-continuous. In particular, ξ_t does not jump at a fixed time, and it is easy to see that ξ_t does not jump at A_{τ_n} . This implies that the radial part r_t and the angular part $\theta_t = \xi_{A_t}$ of x_t do not jump simultaneously. \square

Remark 1 Note that an isotropic self-similar Markov process may not satisfy the condition in Theorem 1. The most famous examples are the symmetric $(1/\alpha)$ -stable Lévy processes for $\alpha > 1/2$. Their Lévy measures are absolutely continuous on $\mathbb{R}^d - \{0\}$, so their radial and angular parts may jump together, and thus do not possess a skew product structure as defined above. On the other hand, we will see later that there are many isotropic α -s.s. Markov processes that do possess a skew product structure.

Remark 2 It is evident that the proof above is also valid for a general isotropic Markov process. That is, the jump condition in Theorem 1 is also necessary for an isotropic Markov process to have a skew product structure.

The rest of this paper is devoted to proving the sufficiency part of Theorem 1. In Section 2, we will recall the time change used in [4], and we will show that x_t is α -s.s. if and only if the time changed process is invariant under the scalar multiplication. The key fact is that if x_t is α -s.s., then the time changed process is invariant under a transitive group on E and

hence may be viewed as an invariant Markov process in a homogeneous space. Under this viewpoint, we complete the proof of our main theorem in Section 3.

2 Time changed processes

Let x_t be an isotropic Markov process (not necessarily α -s.s.) in $E = \mathbb{R}^d - \{0\}$ with a possibly finite life time ξ . Fix $\alpha > 0$. The following random time change was introduced by Lamperti [8] for \mathbb{R}_+ -valued processes. Define

$$A_t = \int_0^t |x_s|^{-1/\alpha} ds, \quad (3)$$

which is a continuous and strictly increasing function for $t < \xi$. Its inverse T_t is given by

$$T_t = \inf\{s \geq 0; A_s \geq t\}, \quad t < A_{\xi-}. \quad (4)$$

We define a new process $\{\bar{x}_t\}$ by $\bar{x}_t = x_{T_t}$ for $t < A_{\xi-}$ and $x_t = \Delta$ otherwise, where Δ is a cemetery point added to E . By Theorem 10.11 of [2], \bar{x}_t is also a time homogeneous Markov process with càdlàg paths. Let $\bar{P}_t(x, B)$ be the transition function of \bar{x}_t . Note that \bar{x}_t and \bar{P}_t are also isotropic.

It is easy to show that

$$T_t = \int_0^t |\bar{x}_u|^{1/\alpha} du \quad (5)$$

for $t < A_{\xi-}$. That is, T_t is determined by the time changed process \bar{x}_t and is also continuous and strictly increasing. Note that A_t is the inverse of T_t . Thus we may start with an isotropic Markov process \bar{x}_t in E and recover the original process x_t as \bar{x}_{A_t} .

The process \bar{x}_t is said to be *multiplicatively invariant*, if

$$\bar{P}_t(x, B) = \bar{P}_t(\lambda x, \lambda B) \quad (6)$$

for any $\lambda > 0$, $x \in E$ and Borel subset $B \subset E$. This is equivalent to saying that process $\lambda \bar{x}_t$ with $\bar{x}_0 = x$ has the same distribution as process \bar{x}_t with $\bar{x}_0 = \lambda x$.

The following theorem relates the α -self-similarity of x_t to the multiplicative invariance of \bar{x}_t . The multiplicative invariance of \bar{x}_t was proved by Kiu [7], but the present proof is simpler and more probabilistic, and also establishes its converse.

Theorem 2 *The process x_t is α -s.s. if and only if the time changed process \bar{x}_t is multiplicatively invariant.*

Proof For simplicity, we will work on the canonical probability space of càdlàg paths with possibly finite life time. We will also write x . for a path x_t in E and x_λ . for path $t \mapsto x_{\lambda t}$ for

$\lambda > 0$. To indicate the dependence on a path x ., we will write $A_t(x)$ and $T_t(x)$ instead of A_t and T_t .

Assume that x_t is α -s.s.. Then $(\lambda^\alpha x_{\lambda^{-1}t}, P_{\lambda^{-\alpha}x})$ is the same Markov process as (x_t, P_x) and consequently, under $P_{\lambda^{-\alpha}x}$, the distribution of $(\lambda^\alpha x_{\lambda^{-1}t}, T_t(\lambda^\alpha x_{\lambda^{-1}}))$ equals that of $(x_t, T_t(x))$ under P_x . Since

$$A_t(\lambda^\alpha x_{\lambda^{-1}}) = \lambda^{-1} \int_0^t |x_{\lambda^{-1}s}|^{-1/\alpha} ds = \int_0^{\lambda^{-1}t} |x_s|^{-1/\alpha} ds = A_{\lambda^{-1}t}(x),$$

we obtain that $T_t(\lambda^\alpha x_{\lambda^{-1}}) = \lambda T_t(x)$. Note that the processes $\lambda^\alpha \bar{x}_t$ and \bar{x}_t are respectively measurable functionals of the processes $(\lambda^\alpha x_{\lambda^{-1}t}, \lambda T_t(x))$ and $(x_t, T_t(x))$ of the same form. It follows that process $\lambda^\alpha \bar{x}_t$ with $\bar{x}_0 = \lambda^{-\alpha}x$ has the same distribution as process \bar{x}_t with $\bar{x}_0 = x$. This proves the multiplicative invariance of \bar{x}_t .

Conversely, assume that \bar{x}_t is multiplicatively invariant. Then the process $\lambda^\alpha \bar{x}_t$ with $\bar{x}_0 = x$ has the same distribution as the process \bar{x}_t with $\bar{x}_0 = \lambda^\alpha x$. Let $\bar{T}_t(\bar{x})$ denote the integral in (5) and let $\bar{A}_t(\bar{x})$ be its inverse as a function of t . Then $A_t(x) = \bar{A}_t(\bar{x})$, and the distribution of $(\lambda^\alpha \bar{x}_t, \bar{A}_t(\lambda^\alpha \bar{x}))$ with $\bar{x}_0 = x$ equals that of $(\bar{x}_t, \bar{A}_t(\bar{x}))$ with $\bar{x}_0 = \lambda^\alpha x$. Because $\bar{T}_t(\lambda^\alpha \bar{x}) = \lambda \bar{T}_t(\bar{x})$, $\bar{A}_t(\lambda^\alpha \bar{x}) = \bar{A}_{\lambda^{-1}t}(\bar{x}) = A_{\lambda^{-1}t}(x)$. The α -self-similarity of x_t now follows from a substitution of $\bar{A}_t(\lambda^\alpha \bar{x})$ for t in $\lambda^\alpha \bar{x}_t$. \square

As in [4], the semigroup property implies that there is a $\gamma \geq 0$ such that $\bar{P}_t(x, E) = e^{-\gamma t}$ for $t \geq 0$ and $x \in E$. When $\gamma > 0$, \bar{x}_t will have a finite life time, or equivalently, \bar{P}_t is not conservative. But we may define a new transition function \hat{P}_t by

$$\hat{P}_t(x, B) = e^{\gamma t} \bar{P}_t(x, B), \quad t \geq 0, \quad x \in E, \quad B \subset E.$$

Then \hat{P}_t is a conservative transition function, and the associated conservative Markov process \hat{x}_t is isotropic and multiplicatively invariant. The process \bar{x}_t is just process \hat{x}_t killed at an independent exponential time of rate γ .

3 Proof of the sufficiency part in Theorem 1

Let $d \geq 2$ and let $GL(d, \mathbb{R})$ be the group of the nonsingular linear transformations on \mathbb{R}^d . Let G be the similarity group of \mathbb{R}^d , that is,

$$G = \{g \in GL(d, \mathbb{R}); |gv| = |g||v| \text{ for any } v \in \mathbb{R}^d\},$$

where $|v| = \sqrt{v_1^2 + \cdots + v_d^2}$ for $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ and $|g|$ is the operator norm of $g \in GL(d, \mathbb{R})$, that is, $|g| = \sup_{|v|=1} |gv|$. For $c > 0$, define the linear transformation m_c by $m_c v = cv$ for $v \in \mathbb{R}^d$. Let $R = \{m_c; c > 0\}$ and $H = O(d)$. Then R and H are both normal subgroups of G . Moreover, G is the direct product of R and H .

Note that G acts transitively on $E = \mathbb{R}^d - \{0\}$. Fix $o = (0, \dots, 0, 1)$. The subgroup of G fixing o is $K = O(d-1)$. We may identify G/K with E via the map $gK \mapsto go$, H/K with the sphere S^{d-1} via $hK \mapsto ho$, and R with a ray in $E \subset \mathbb{R}^d$ via $r \mapsto ro$. Note that E is diffeomorphic to the product space $R \times S^{d-1}$.

The reader is referred to section 2.2 of [9] for some basic definitions about invariant Markov processes in homogeneous spaces. Let \mathfrak{g} , \mathfrak{r} , \mathfrak{h} and \mathfrak{k} be respectively the Lie algebras of G , R , H and K . There is an $\text{Ad}(K)$ -invariant subspace \mathfrak{p} such that $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$. Then the exponential map of G provides a natural local diffeomorphism from $\mathfrak{r} \oplus \mathfrak{p}$ to E . Let $n = \dim G$ and let $\{X_1, \dots, X_n\}$ be a basis of \mathfrak{g} such that $X_1 \in \mathfrak{r}$, $X_2, \dots, X_d \in \mathfrak{p}$ and $X_{d+1}, \dots, X_n \in \mathfrak{k}$. Let $\pi: G \rightarrow E$ be the map $g \mapsto go$. Restricted to a sufficient small neighborhood V of 0, the map

$$\phi: \mathbb{R}^d \ni y = (y_1, \dots, y_d) \mapsto \pi(e^{\sum_{j=1}^d y_j X_j}) \in E$$

is a diffeomorphism and y_1, \dots, y_d may be used as local coordinates on $\phi(V)$. As in Section 2.2 of Liao [9], we may extend y_j to E such that $y_j \in C_c^\infty(E)$ (the space of smooth functions on E with compact supports) and for any $x \in E$, $k \in K$,

$$\sum_{j=1}^d y_j(x) \text{Ad}(k)X_j = \sum_{j=1}^d y_j(kx)X_j. \quad (7)$$

As in Section 2, we let x_t be an isotropic α -s.s. Markov process starting at $x \in E$. Recall that the time changed process \bar{x}_t defined before is a G -invariant Markov process in E with transition function \bar{P}_t (see Theorem 2). Thus for any $f \in C_c^\infty(E)$ and $x \in E$,

$$\bar{P}_t f(x) = \bar{P}_t(f \circ g)(o),$$

where $g \in G$ is chosen to satisfy $x = go$. As an easy consequence, \bar{P}_t is a G -invariant Feller semigroup on E .

Let L be the generator of \bar{x}_t with domain $\text{Dom}(L)$. An explicit formula for the generator of an invariant Markov process in a homogeneous space was obtained by Hunt [5]. By Theorem 2.1 of [9], which is a more convenient version of Hunt's formula, $\text{Dom}(L)$ contains $C_c^\infty(E)$ and for $f \in C_c^\infty(E)$,

$$Lf(o) = Tf(o) + \int_E \left[f(x) - f(o) - \sum_{j=1}^d y_j(x) \frac{\partial}{\partial y_j} f(o) \right] \Pi(dx), \quad (8)$$

where T is a G -invariant diffusion generator and Π is a K -invariant Lévy measure on E . There exist a $d \times d$ non-negative definite symmetric matrix (a_{ij}) and constants c_i such that for $f \in C_c^\infty(E)$,

$$Tf(o) = \frac{1}{2} \sum_{i,j=1}^d a_{ij} X_i^l X_j^l (f \circ \pi)(e) + \sum_{i=1}^d c_i X_i^l (f \circ \pi)(e), \quad (9)$$

where X_i^l is the left invariant vector field on G determined by X_i . Moreover, the coefficients a_{ij} and c_i satisfy

$$a_{ij} = \sum_{p,q=1}^d a_{pq} b_{ip}(k) b_{jq}(k) \quad \text{and} \quad c_i = \sum_{p=1}^d c_p b_{ip}(k), \quad \forall k \in K, \quad (10)$$

where the orthogonal matrix $(b_{ij}(k))$ is determined by $\text{Ad}(k)X_j = \sum_{i=1}^d b_{ij}(k)X_i$ for $j = 1, \dots, d$.

Since R commutes with H and \mathfrak{p} is $\text{Ad}(K)$ -invariant, $\text{Ad}(k)X_1 = X_1$ and $\text{Ad}(k)X_i \in \mathfrak{p}$ for $i \geq 2$ and $k \in K$. Thus $b_{11}(k) = 1$, $b_{i1}(k) = b_{1i}(k) = 0$ for $i \geq 2$. Then (10) implies that

$$a_{i1} = \sum_{p=2}^d a_{p1} b_{ip}(k), \quad a_{1i} = \sum_{q=2}^d a_{1q} b_{iq}(k), \quad \text{and} \quad c_i = \sum_{p=2}^d c_p b_{ip}(k), \quad 2 \leq i \leq d.$$

In other words, the vectors $X = \sum_{i=2}^d a_{i1}X_i$ and $Y = \sum_{i=2}^d a_{1i}X_i$ are invariant under the action of $\text{Ad}(k)$ for all $k \in K$, which implies that $X = Y = 0$. Hence $a_{i1} = a_{1i} = 0$ for $2 \leq i \leq d$. The operator T_2 defined by

$$T_2 f(o) = \frac{1}{2} \sum_{i,j=2}^d a_{ij} X_i^l X_j^l (f \circ \pi)(e) + \sum_{i=2}^d c_i X_i^l (f \circ \pi)(e), \quad f \in C_c^\infty(E)$$

may be viewed as an H -invariant diffusion generator on the sphere $S^{d-1} = H/K$. It is well known that there is a constant $c \geq 0$ such that $T_2 = c\Delta$, where Δ is the Laplace-Beltrami operator on S^{d-1} . We define the diffusion generator T_1 by

$$T_1 f(o) = \frac{1}{2} a_{11} X_1^l X_1^l (f \circ \pi)(e) + c_1 X_1^l (f \circ \pi)(e), \quad f \in C_c^\infty(E).$$

Note that operator T_1 acts along R . We have proved that $T = T_1 + T_2$ in the sense that

$$Tf(r, \theta) = (T_1 f(\cdot, \theta))(r) + (T_2 f(r, \cdot))(\theta)$$

for $r \in R$ and $\theta \in S^{d-1}$.

Let π_1 (resp. π_2) be the projection from E onto R (resp. S^{d-1}). Then for $x \in E$, $\pi_1(x)$ (resp. $\pi_2(x)$) may be identified with $|x|$ (resp. $x/|x|$). Let $\rho_t = \pi_1(\bar{x}_t)$ and $\xi_t = \pi_2(\bar{x}_t)$. By the $O(d)$ -invariance of \bar{x}_t , ρ_t is a Lévy process on R starting at $\pi_1(x)$ and ξ_t is an $O(d)$ -invariant Feller process on the sphere S^{d-1} starting at $\pi_2(x)$.

Proposition 1 *ρ_t and ξ_t are independent if and only if the Lévy measure Π of \bar{x}_t is concentrated on $R \cup S^{d-1}$, where R and S^{d-1} are regarded as subsets of $E \cong R \times S^{d-1}$.*

Proof Assume that ρ_t and ξ_t are independent. Let f_1 (resp. f_2) be a smooth function on R (resp. S^{d-1}) vanishing near o . Let $f(x) = f_1(\pi_1(x))f_2(\pi_2(x))$. Then by (8), $\Pi(f) = \int_E f(x)\Pi(dx) = Lf(o)$. From the independence of ρ_t and ξ_t , we have that

$$Lf(o) = \lim_{t \rightarrow 0} \frac{E[f(\bar{x}_t)]}{t} = \lim_{t \rightarrow 0} \frac{E[f_1(\rho_t)]E[f_2(\xi_t)]}{t}.$$

It follows that $\Pi(f) = 0$ since $E[f_1(\rho_t)] = tO(t)$ and $E[f_2(\xi_t)] = tO(t)$ as $t \rightarrow 0$.

Now fix a point $x = (r, \theta) \in E$ such that r and θ are not the point o . We may choose positive functions f_1 on R and f_2 on S^{d-1} satisfying the above conditions and additionally, we assume that $f_1 = 1$ near r and that $f_2 = 1$ near θ . Then there exists a neighborhood U of (r, θ) such that $\Pi(U) \leq \Pi(f_1 f_2) = 0$. Hence (r, θ) is not contained in the support of Π . It follows that $\text{supp}\Pi \subset R \cup S^{d-1}$.

Conversely, let $\Pi = \Pi_1 + \Pi_2$ be such that Π_1 and Π_2 are respectively Lévy measures on R and S^{d-1} , regarded as measures on E supported by R and S^{d-1} . For $i = 1, 2$, let L_i be generators with diffusion parts T_i and Lévy measures Π_i . Our computation shows that $L = L_1 + L_2$ at point o , and by the G -invariance of the three operators, $L = L_1 + L_2$ on E . Note that when restricted to R (resp. S^{d-1}), L_1 (resp. L_2) is the generator of ρ_t (resp. ξ_t). Let $\tilde{\rho}_t$ be a Lévy process in R with generator L_1 and let $\tilde{\xi}_t$ be an $O(d)$ -invariant Markov process in S^{d-1} with generator L_2 , and let them be independent. Then $\tilde{x}_t = (\tilde{\rho}_t, \tilde{\xi}_t)$ is a G -invariant Markov process in E with generator $L = L_1 + L_2$. By the uniqueness in Theorem 2.1 of [9], the processes $\tilde{x}_t = (\tilde{\rho}_t, \tilde{\xi}_t)$ and $\bar{x}_t = (\rho_t, \xi_t)$ have the same distribution because they have the same generator. This shows that ρ_t and ξ_t are independent. \square

Proof of sufficiency in Theorem 1 Now we assume that the radial and angular parts of x_t do not jump at same time. It is obvious that the time change $\bar{x}_t = x_{T_t}$ does not change the directions of jumps. Thus the Lévy measure of \bar{x}_t is concentrated on the radial and angular axes. By Proposition 1, $\rho_t = \pi_1(\bar{x}_t)$ and $\xi_t = \pi_2(\bar{x}_t)$ are independent, and $\bar{x}_t = \rho_t \xi_t$. Recall that $A_t = \int_0^t |x_s|^{-1/\alpha} ds$ is the inverse of $T_t = \int_0^t (\rho_s)^{1/\alpha} ds$ and $x_t = \bar{x}_{A_t}$. Then $x_t = r_t \xi_{A_t}$, where $r_t = |x_t| = \rho_{A_t}$ is an α -self-similar process on $(0, \infty)$ and is independent of ξ_t . Thus the skew product structure of x_t is established. \square

Remark 3 Our proof shows that the time change by A_t provides a 1-1 correspondence between isotropic α -s.s. Markov processes and G -invariant Markov processes in E . Thus, given any Lévy measure supported by $R \cup S^{d-1}$, there is a unique isotropic α -s.s. Markov process in E that possesses a skew product structure. We also note that any isotropic α -s.s. Markov process has the strong Markov property, because the strong Markov property is possessed by the time changed process and is preserved by the inverse time change.

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